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Kähler quantization of $H^3(CY_3, R)$ and the holomorphic anomaly

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Abstract: Studying the quadratic field theory on seven dimensional spacetime constructed by a direct product of Calabi-Yau three-fold by a real time axis, with phase space being the third cohomology of the Calabi-Yau three-fold, the generators of translation along moduli directions of Calabi-Yau three-fold are constructed. The algebra of these generators is derived which take a simple form in canonical coordinates. Applying the Dirac method of quantization of second class constraint systems, we show that the Schrödinger equations corresponding to these generators are equivalent to the holomorphic anomaly equations if one defines the action functional of the quadratic field theory with a proper factor one-half.

Keywords: Models of Quantum Gravity, Topological Strings, Field Theories in Higher Dimensions.

Contents

1. Introduction and summary

In [1, 2], Bershadsky, Cecotti, Ouguri and Vafa described a holomorphic anomaly in topological string theories obtained by twisting $N = 2$ models. They described this anomaly as a subtle breakdown of the BRST invariance in the twisted N=2 model coupled to gravity. In [3] Witten discussed the implication of the holomorphic anomaly to the background independence of the string theory. He showed that the holomorphic anomaly can be understood as a violation of the naive background independence that explains some general puzzles in mirror symmetry as a relation between A-model and B-model of Calabi-Yau manifolds. He also derived two equations resembling the holomorphic anomaly equations derived in [1] by quantizing $H^3(CY_3, R)$, see also [4, 5]. This observation is one of the most straightforward realizations of the idea that the partition function of the type B topological strings on Calabi-yau manifolds is related to a holomorphic wave function in the seven dimensional theory with a phase space being the $H^3(CY_3, R)$ [4].

In this paper we consider the quadratic field theory on seven dimensional spacetime $CY_3 \times R$, given by the action,

$$S(C) = \text{const.} \int_{CY_3 \times R} C \wedge dC\,, \tag{1.1}$$

The phase space of this theory is known to be $H^3(CY_3, R)$ [4, 5]. We derive the explicit form of H_i and $\bar{H}_{\bar{i}}$, the generators of translation along moduli directions t_i and $\bar{t}_{\bar{i}}$ of the Calabi-Yau three-fold. We show that in canonical coordinates where $\bar{H}_{\bar{i}} = \overline{H_i}$, these generators satisfy the algebra,

$$\begin{aligned} \{\bar{H}_{\bar{i}}, \bar{H}_{\bar{j}}\} &= 0\,, \\ \{H_i, H_j\} &= 0\,, \\ \{H_i, \bar{H}_{\bar{i}}\} &= \partial_i \bar{H}_{\bar{i}} - \bar{\partial}_{\bar{i}} H_i\,. \end{aligned} \tag{1.2}$$

After quantization it is shown that, if the constant factor in eq. (1.1) is equal to one-half, i.e. for the action,

$$S(C) = \frac{1}{2}\int_{CY_3\times R} C\wedge dC\,, \tag{1.3}$$

the Schrödinger equations

$$i\partial_i\left|\psi\right\rangle = H_i\left|\psi\right\rangle, \qquad i\bar{\partial}_{\bar{i}}\left|\psi\right\rangle = \bar{H}_{\bar{i}}\left|\psi\right\rangle, \tag{1.4}$$

are equivalent to holomorphic anomaly equations in the simplified form given in [4].

The organization of the paper is as follows. In section 2, the classical theory on $CY_3\times R$ is studied. The generators H_i and $\bar{H}_{\bar{i}}$ are constructed and their algebra is studied. In section 3, it is shown that the Schrödinger equations with respect to generators H_i and $\bar{H}_{\bar{i}}$ are equivalent to the holomorphic anomaly equations in topological string theory. Finally we determine the true constant factor in definition of the action functional. For this purpose we first study the constraint structure of the theory and derive Dirac brackets in section 3.1. Using these results the factor one-half in eq. (1.3) is derived in section 3.2.

2. The Calabi-Yau moduli space

A Calabi-Yau manifold (CY) is a compact Kähler manifold with vanishing first Chern class. According to a conjecture by Calabi, proven by Yau, CY manifolds admit a Ricci-flat metric. Thus CY manifolds provide solutions to the Einstein equation. Calabi-Yau three-fold (CY_3) is considered for superstring compactification.

The vanishing of the first Chern class implies that the canonical bundle is trivial. Thus in the case of CY_3, the holomorphic three-form $\Omega\in H^{3,0}(CY_3,R)$ (the CY-form) is unique only up to a scale. By deformations of the complex structure, a $(3,0)$-form can only change into a linear combination of $(3,0)$-form and $(2,1)$-forms, thus $H^{2,1}$ measures complex structure deformation. In fact, H^3 forms a bundle over the moduli space of complex structures of CY_3 and the CY-form Ω defines a line sub-bundle $\mathcal{L}$. A Kähler potential can be defined by the relation

$$K = -\ln i\int\bar{\Omega}\wedge\Omega\,, \tag{2.1}$$

which transforms under $\Omega\to e^{f(z)}\Omega$ as $K\to K-f-\bar{f}$. On the line bundle $\mathcal{L}$ one can define a connection $\nabla_i = \partial_i+\partial_i K$ which curvature $G_{ij}=\partial_i\bar{\partial}_{\bar{j}}K$ is the Kähler metric on the moduli space. It is a simple exercise to show that $\nabla_i\Omega$ is the $H^{2,1}$ part of $\partial_i\Omega$.

A real closed $\gamma\in H^3(CY_3,R)$ can be decomposed in the basis consisting the holomorphic three-form $\Omega\in H^{3,0}$, its covariant derivatives $\nabla_i\Omega\in H^{2,1}$ and their complex conjugates,

$$\gamma = \lambda^{-1}\Omega + x^i\nabla_i\Omega + \bar{x}^{\bar{i}}\bar{\nabla}_{\bar{i}}\bar{\Omega} + \bar{\lambda}^{-1}\bar{\Omega}\,. \tag{2.2}$$

γ can be considered as the classical solution of the Euler-Lagrange equation of motion in the seven dimensional field theory on $CY_3\times R$ with action functional [4, 5],

$$S(C) = \int_{CY_3\times R} C\wedge dC\,, \tag{2.3}$$

where C is a real three-form. Using the decomposition $C = \gamma + \omega dt$ in terms of γ a three-form component of C along CY_3 and ωdt a two-form along CY_3 and a one form along R, the action can be written as,

$$S(\gamma, \omega) = \int dt \int_{CY_3} \left(\gamma \frac{\partial}{\partial t} \gamma + \omega d\gamma \right), \tag{2.4}$$

which equation of motion imply that γ is closed and time independent, see section 3.1. Thus the action functional (2.3) can be used to quantize $H^3(CY_3)$. The phase space can be constructed using the coordinates λ^{-1}, x^i's defined in eq. (2.2) and their conjugate momenta defined by the action (2.3),

$$p_i = i e^{-K} G_{i\bar{i}} \bar{x}^{\bar{i}}, \qquad \pi = -i e^{-K} \bar{\lambda}^{-1}, \tag{2.5}$$

with Poisson brackets,

$$\{x^i, p_j\} = \delta^i_j, \qquad \{\lambda^{-1}, \pi\} = 1, \tag{2.6}$$

with all other Poisson brackets vanishing. We assume that the phase space operators x^i, p_j, λ^{-1} and π do not explicitly depend on the moduli $t\bar{t}$. Defining H_i and $\bar{H}_{\bar{i}}$ to be generators of translation along moduli directions t_i and $\bar{t}_{\bar{i}}$ respectively, one has,

$$\begin{aligned} \mathcal{O}_{,i} &= \{\mathcal{O}, H_i\}, \\ \mathcal{O}_{,\bar{i}} &= \{\mathcal{O}, \bar{H}_{\bar{i}}\}. \end{aligned} \tag{2.7}$$

We use the convention $\mathcal{O}_{,i} = \partial_i \mathcal{O} = \frac{\partial}{\partial t_i} \mathcal{O}$ occasionally. To obtain H_i and $\bar{H}_{\bar{i}}$ we note that γ defined in eq. (2.2) is independent of moduli. For example,

$$\begin{aligned} \gamma_{,\bar{i}} &= \lambda^{-1}_{,\bar{i}} \Omega + x^i_{,\bar{i}} \nabla_i \Omega + x^i \bar{\partial}_{\bar{i}} \nabla_i \Omega + \bar{x}^{\bar{j}}_{,\bar{i}} \bar{\nabla}_{\bar{j}} \bar{\Omega} + \bar{x}^{\bar{j}} \bar{\partial}_{\bar{i}} \bar{\nabla}_{\bar{j}} \bar{\Omega} + \bar{\lambda}^{-1}_{,\bar{i}} \bar{\Omega} + \bar{\lambda}^{-1} \bar{\partial}_{\bar{i}} \bar{\Omega} \\ &= (\lambda^{-1}_{,\bar{i}} + x^i G_{i\bar{i}}) \Omega + (x^k_{,\bar{i}} - e^K \bar{x}^{\bar{j}} G^{k\bar{k}} \bar{C}_{\bar{i}\bar{j}\bar{k}}) \nabla_k \Omega + (\bar{x}^{\bar{k}}_{,\bar{i}} + \bar{\lambda}^{-1} \delta^{\bar{k}}_{\bar{i}} - \bar{\partial}_{\bar{i}} K - \bar{x}^{\bar{j}} \Gamma^{\bar{k}}_{\bar{i}\bar{j}}) \bar{\nabla}_{\bar{k}} \bar{\Omega} + \\ &\quad + (\bar{\lambda}^{-1}_{,\bar{i}} - \bar{\lambda}^{-1} \bar{\partial}_{\bar{i}} K) \bar{\Omega} \\ &= 0. \end{aligned} \tag{2.8}$$

To obtain the second equality above we have used the identities (2.10) that can be verified using the following equations,

$$\begin{aligned} e^{-K} &= i \int_{CY_3} \bar{\Omega} \wedge \Omega, \\ e^{-K} G_{i\bar{j}} &= i \int_{CY_3} \nabla_i \Omega \wedge \bar{\nabla}_{\bar{j}} \bar{\Omega}, \\ C_{ijk} &= i \int_{CY_3} \nabla_i \Omega \wedge D_i \nabla_k \Omega, \end{aligned} \tag{2.9}$$

where C_{ijk} is the three point function in the B-model, $C_{ijk} = -i \int_{CY_3} \Omega \wedge \partial_i \partial_j \partial_k \Omega$. The covariant derivative D_i in eq. (2.9) contains the usual Christoffel connection $\Gamma^k_{ij} = -G^{k\bar{k}} G_{k\bar{j},\bar{i}}$ as well as the term $\partial_i K$ see for example eq. (2.15). Eq. (2.9) implies that modulo exact terms one has,

$$\bar{\partial}_{\bar{i}} \nabla_j \Omega = G_{\bar{i}j} \Omega, \qquad D_i \nabla_j \Omega = -e^K G^{k\bar{k}} C_{ijk} \bar{\nabla}_{\bar{k}} \bar{\Omega}. \tag{2.10}$$

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From eq. (2.8) and its complex conjugate ($\gamma_{,i}=0$) one obtains,

$$\begin{aligned}
\lambda^{-1}_{,\bar i} &= -G_{\bar i i}x^i\,, & \bar\lambda^{-1}_{,i} &= -G_{i\bar i}\bar x^{\bar i}\,,\\
x^k_{,\bar i} &= e^K G^{k\bar k}\bar C_{\bar i\bar j\bar k}\bar x^{\bar j}\,, & \bar x^{\bar k}_{,i} &= e^K G^{\bar k k}C_{ijk}x^j\,,\\
\bar x^{\bar k}_{,\bar i} &= -\bar\lambda^{-1}\delta^{\bar k}_{\bar i} + \bar x^{\bar j}\bar\Gamma^{\bar k}_{\bar i\bar j} + \bar\partial_{\bar i}K\bar x^{\bar k}\,, & x^k_{,i} &= -\lambda^{-1}\delta^k_i + x^j\Gamma^k_{ij} + \partial_i K x^k\,,\\
\bar\lambda^{-1}_{,\bar i} &= \bar\lambda^{-1}\bar\partial_{\bar i}K\,, & \lambda^{-1}_{,i} &= \lambda^{-1}\partial_i K\,.
\end{aligned} \tag{2.11}$$

It is straightforward to verify that using eqs. (2.5), (2.6) and (2.7) one can obtain eq. (2.11) if

$$\begin{aligned}
H_i &= -\lambda^{-1}p_i - \frac{i}{2}C_{ijk}x^j x^k + \frac{\partial_i K}{2}[\pi,\lambda^{-1}]_+ + \frac{1}{2}\Gamma^k_{ij}[x^j,p_k]_+ + \frac{\partial_i K}{2}[x^j,p_j]_+,\\
\bar H_{\bar i} &= -G_{\bar i i}\pi x^i - \frac{i}{2}e^{2K}\bar C_{\bar i\bar j\bar k}G^{\bar j j}G^{\bar k k}p_j p_k,
\end{aligned} \tag{2.12}$$

where we have used the notation $[A,B]_+ = AB+BA$ for later convenience. At the classical level $[A,B]_+ = 2AB$ as far as there is no operator ordering problem. For example to obtain the equality $\bar\lambda^{-1}_{,i} = -G_{i\bar i}\bar x^{\bar i}$ given in eq. (2.11), one uses eqs. (2.7) and (2.12) to obtain $\pi_{,i} = p_i - K_{,i}\pi$. Then using the relation $\pi = -ie^{-K}\bar\lambda^{-1}$ given in eq. (2.5) to calculate $\bar\lambda^{-1}_{,i}$ in terms of $\pi_{,i}$, one obtains the desired result.

The algebra of H_i and $\bar H_{\bar i}$ is interesting. It is easy to verify that

$$\{\bar H_{\bar i},\bar H_{\bar j}\} = 0\,. \tag{2.13}$$

After some calculations one can show that

$$\begin{aligned}
\{H_i,H_j\} &= i\left[\left(C_{jlk}\Gamma^k_{in} + C_{jln}\partial_i K\right) - i\leftrightarrow j\right]x^l x^n + \left(\Gamma^k_{im}\Gamma^m_{jl} - i\leftrightarrow j\right)x^l p_k\\
&= i\left[(D_i-\partial_i)C_{jnl} - i\leftrightarrow j\right]x^l x^n - \left(\partial_i\Gamma^k_{jl} - i\leftrightarrow j\right)x^l p_k\\
&= i\left(\partial_j C_i - \partial_i C_j\right)_{nl}x^l x^n + \left(\partial_j\Gamma^k_{il} - \partial_i\Gamma^k_{jl}\right)x^l p_k\,.
\end{aligned} \tag{2.14}$$

The connection D_i is defined by the relation,

$$D_i C_j = \partial_i C_j + \partial_i K C_j + \Gamma^k_{ij}C_k\,, \tag{2.15}$$

The second and third equalities in eq. (2.14) are obtained using the fact that the four point function $C_{ijkl} = D_i C_{jkl}$ is totally symmetric in its four indices [1], which is the consequence of the tt^* equation $D_i C_j = D_j C_i$ [6]. We have also used the identity,

$$R^k_{jil} = \left(\partial_i\Gamma^k_{jl} + \Gamma^k_{im}\Gamma^m_{jl}\right) - i\leftrightarrow j = 0\,. \tag{2.16}$$

Eq. (2.14) can be more simplified using the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equation $\partial_i C_j = \partial_j C_i$ to obtain,

$$\{H_i,H_j\} = \partial_j H_i - \partial_i H_j\,. \tag{2.17}$$

Finally one can show that

$$\{H_i,\bar H_{\bar i}\} = \partial_i\bar H_{\bar i} - \bar\partial_{\bar i}H_i\,. \tag{2.18}$$

To obtain the above equality we have used the identities,

$$\begin{aligned}
\partial_i \bar{C}_{\bar{i}\bar{j}\bar{k}} &= \bar{\partial}_{\bar{i}} C_{ijk} = 0\,, \\
R^l_{i\bar{j}k} = -\bar{\partial}_{\bar{j}} \Gamma^l_{ki} &= G_{k\bar{j}}\delta^l_i + G_{i\bar{j}}\delta^l_k - e^{2k} C_{ikn} G^{n\bar{n}} \bar{C}_{\bar{j}\bar{m}\bar{n}} G^{\bar{m}l}\,,
\end{aligned} \tag{2.19}$$

which result in,

$$\begin{aligned}
-\bar{\partial}_{\bar{i}} H_i &= -G_{i\bar{i}}\left(\lambda^{-1}\pi + x^k p_k\right) + R^l_{i\bar{i}k} x^k p_l\,, \\
-\partial_i \bar{H}_{\bar{i}} &= -\pi \Gamma^k_{ij} G_{k\bar{i}} x^j + i e^{2K} G^{j\bar{j}} G^{l\bar{k}} \bar{C}_{\bar{i}\bar{j}\bar{k}} \Gamma^l_{ij} p_j p_l + i\partial_i K e^{2K} G^{j\bar{j}} G^{l\bar{k}} \bar{C}_{\bar{i}\bar{j}\bar{k}} p_j p_k\,.
\end{aligned} \tag{2.20}$$

More interesting relations can be obtained in canonical coordinates. In canonical coordinates $\partial_i K$ and Γ^k_{ij} together with all holomorphic derivatives are vanishing locally. Thus in these coordinates, the right hand side of the first equality in eq. (2.14) is simply vanishing. Consequently, at least locally,

$$\begin{aligned}
\{\bar{H}_{\bar{i}}, \bar{H}_{\bar{j}}\} &= 0\,, \\
\{H_i, H_j\} &= 0\,, \\
\{H_i, \bar{H}_{\bar{i}}\} &= \partial_i \bar{H}_{\bar{i}} - \bar{\partial}_{\bar{i}} H_i\,.
\end{aligned} \tag{2.21}$$

Furthermore using the definitions (2.5) and (2.12) one obtains

$$\bar{H}_{\bar{i}} = \overline{H_i}\,. \tag{2.22}$$

3. Quantization and the holomorphic anomaly

For quantization we consider, as usual, the phase space coordinates as operators satisfying the commutation relations consistent with the Poisson algebra (2.6),

$$[x^i, p_j] = i\delta^i_j\,, \qquad [\lambda^{-1}, \pi] = i, \tag{3.1}$$

with all other commutators vanishing. Quantum states satisfy the set of Schrödinger equations,

$$\begin{aligned}
i\partial_i |\psi\rangle &= H_i |\psi\rangle\,, \\
i\bar{\partial}_{\bar{i}} |\psi\rangle &= \bar{H}_{\bar{i}} |\psi\rangle\,.
\end{aligned} \tag{3.2}$$

Using the commutation relations (3.1), the generator H_i defined in eq. (2.12) can be written in a simple form,

$$H_i = -\lambda^{-1} p_i - \frac{i}{2} C_{ijk} x^j x^k - \partial_i K \left(-\lambda^{-1}\pi - x^j p_j + i\frac{h+1}{2}\right) + \Gamma^k_{ij} x^j p_k - \frac{i}{2} \ln |G|_{,i}\ , \tag{3.3}$$

where we have used the identity $\ln |G|_{,i} = \Gamma^j_{ij}$ and $h = h^{2,1}$. In (x, λ^{-1}) space, where $p_i = -i\frac{\partial}{\partial x^i}$ and $\pi = -i\frac{\partial}{\partial \lambda^{-1}}$, the set of Schrödinger equations (3.2) give,

$$\begin{aligned}
\bar{\partial}_{\bar{i}} \Psi &= \left[G_{\bar{i}i} x^i \frac{\partial}{\partial \lambda^{-1}} + \frac{1}{2} e^{2K} \bar{C}_{\bar{i}\bar{j}\bar{k}} G^{\bar{j}j} G^{\bar{k}k} \frac{\partial^2}{\partial x^j \partial x^k}\right] \Psi\,, \\
\left[\nabla_i + \Gamma^k_{ij} x^j \frac{\partial}{\partial x^k}\right] \Psi &= \left[\lambda^{-1} \frac{\partial}{\partial x^i} - \frac{1}{2} \ln |G|_{,i} - \frac{1}{2} C_{ijk} x^j x^k\right] \Psi\,,
\end{aligned} \tag{3.4}$$

where the connection ∇_i is given by

$$\nabla_i = \partial_i + \partial_i K \left(\frac{h+1}{2} + x^j \frac{\partial}{\partial x^j} + \lambda^{-1} \frac{\partial}{\partial \lambda^{-1}} \right). \tag{3.5}$$

Eq. (3.4) is the holomorphic anomaly equation obtained in [4].

3.1 Constraint structure and Dirac brackets

In this section we digress to study the constraint structure of the action (2.3) which is essential to obtain the true coefficient of the action functional (2.3). For an introduction to constraint systems see [7].

The second term in eq. (2.4) introduces a first class constraint $d\gamma = 0$ which makes γ closed. This is a secondary constraint following the primary constraint $\pi_\omega = \frac{\delta S}{\delta \dot\omega} = 0$ where π_ω denotes the momentum conjugate to the three-form ω. Using eq. (2.2) and (2.9), the first term in the action (2.4) in terms of coordinates λ^{-1}, x^i's and their complex conjugates can be written as follows,

$$S(\lambda^{-1}, x^i, \bar\lambda, \bar\lambda^{-1}) = ie^{-K} \left(\lambda^{-1} \dot{\bar\lambda}^{-1} - \dot\lambda^{-1} \bar\lambda^{-1} \right) - ie^{-K} G_{i\bar i} \left(x^i \dot{\bar x}^{\bar i} - \dot x^i \bar x^{\bar i} \right). \tag{3.6}$$

The momenta conjugate to the coordinates λ^{-1}, x^i's and their complex conjugates can be defined using the general rule,

$$\begin{aligned} \pi &= \frac{\delta S}{\delta \dot\lambda^{-1}} = -ie^{-K} \bar\lambda^{-1}, & \bar\pi &= \frac{\delta S}{\delta \dot{\bar\lambda}^{-1}} = ie^{-K} \lambda^{-1}, \\ p_i &= \frac{\delta S}{\delta \dot x^i} = ie^{-K} G_{i\bar i} \bar x^{\bar i}, & \bar p_{\bar i} &= \frac{\delta S}{\delta \dot{\bar x}^{\bar i}} = -ie^{-K} G_{i\bar i} x^i . \end{aligned} \tag{3.7}$$

Since momenta are independent of velocities, the above equations in fact are relations between phase space coordinates, i.e. they are Dirac constraints. Denoting these constraints as χ_I, $I = 1, \cdots, 2h+2$,

$$\chi_I = \begin{cases} \chi_i \,, & I = 1, \ldots, h \\ \bar\chi_{\bar i} \,, & I = h+1, \ldots, 2h \\ \chi_0 \,, & I = 2h+1 \,, \\ \bar\chi_0 & I = 2h+2 \,, \end{cases} \tag{3.8}$$

where,

$$\begin{aligned} \chi_i &= p_i - ie^{-K} G_{i\bar i} \bar x^{\bar i}, & \bar\chi_{\bar i} &= \bar p_{\bar i} + ie^{-K} G_{i\bar i} x^i , \\ \chi_0 &= \pi + ie^{-K} \bar\lambda^{-1}, & \bar\chi_0 &= \bar\pi - ie^{-K} \lambda^{-1}, \end{aligned} \tag{3.9}$$

are constraints defined by eq. (3.7) and using the canonical Poisson brackets,

$$\begin{aligned} \{x^i, p_j\} &= \delta^i_j \,, & \{\bar x^{\bar i}, \bar p_{\bar j}\} &= \delta^{\bar i}_{\bar j} \,, \\ \{\lambda^{-1}, \pi\} &= 1 & \{\bar\lambda^{-1}, \bar\pi\} &= 1 \,, \end{aligned} \tag{3.10}$$

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with all other Poisson brackets vanishing, one verifies that

$$\begin{aligned}
\{\chi_i, \bar{\chi}_{\bar{i}}\} &= -2ie^{-K} G_{i\bar{i}}\,, \\
\{\chi_0, \bar{\chi}_0\} &= 2ie^{-K}\,, \\
\{\chi_0(\bar{\chi}_0), \chi_i\} &= 0\,, \\
\{\chi_0(\bar{\chi}_0), \bar{\chi}_{\bar{i}}\} &= 0\,.
\end{aligned} \tag{3.11}$$

Therefore,

$$\det\left(\{\chi_I, \chi_J\}\right) \neq 0\,, \tag{3.12}$$

which means that the constraints χ_I's are of second class. Since the number of second class constraints is equal to the number of degrees of freedom, by imposing the constraints there remains no degree of freedom with respect to time t in this theory. In this sense the theory is time independent. For quantization one should use Dirac brackets which are defined in terms of the canonical Poisson brackets as follows:

$$\{A, B\}_{\rm DB} = \{A, B\} - \{A, \chi_I\}\chi^{IJ}\{\chi_J, B\}\,, \tag{3.13}$$

where χ^{IJ} is the inverse of the matrix $\chi_{IJ} = \{\chi_I, \chi_J\}$. For example

$$\begin{aligned}
\{\lambda^{-1}, \bar{\lambda}^{-1}\}_{\rm DB} &= \frac{i}{2}e^K\,, \\
\{x^i, \lambda^{-1}(\bar{\lambda}^{-1})\}_{\rm DB} &= 0\,, \\
\{\bar{x}^{\bar{i}}, \lambda^{-1}(\bar{\lambda}^{-1})\}_{\rm DB} &= 0\,, \\
\{x^i, \bar{x}^{\bar{i}}\}_{\rm DB} &= -\frac{i}{2}e^K G^{i\bar{i}}\,,
\end{aligned} \tag{3.14}$$

where in the last equality above, $G^{i\bar{i}}$ is the inverse of the metric $G_{i\bar{i}}$.

This is an example of non-commutativity of coordinates. Comparing the Dirac brackets obtained above with Poisson brackets (3.7) and the constraints (3.9) it is easy to verify that in this theory, one can obtain true commutation relations (i.e. the Dirac brackets) by simply solving the momenta in terms of the coordinates by imposing the constraints $\chi_I = 0$ and inserting the solution into Poisson brackets (3.10). Of course one makes a mistake in evaluating the coefficient (here the factor 1/2) in doing so. What supports the validity of that wrong method (up to a factor 1/2 here) is the fact that by construction by using the Dirac brackets one can safely assume that constraints are solved since using eq. (3.13),

$$\{A, \chi_I\}_{\rm DB} = \{\chi_J, B\}_{\rm DB} = 0\,, \qquad I, J = 1, \ldots, 2h+2\,. \tag{3.15}$$

and consider the *reduced phase space* possessing only coordinates λ^{-1}, x^i and their conjugate momenta π, p_i with Dirac brackets,

$$\begin{aligned}
\{x^i, p_j\}_{\rm DB} &= \frac{1}{2}\delta^i_j\,, \\
\{\lambda^{-1}, \pi\}_{\rm DB} &= \frac{1}{2}\,,
\end{aligned} \tag{3.16}$$

and all other brackets vanishing. The above equations are consistent with the non-commutativity algebra (3.14) and constraints (3.9) as is expected from the identity (3.15). Finally we note that in constraint systems with second class constraints, for quantization the commutators are constructed in terms of the Dirac brackets instead of Poisson brackets. Therefore,

$$p_i \to \frac{-i}{2}\frac{\partial}{\partial x^i}\,, \qquad \pi \to \frac{-i}{2}\frac{\partial}{\partial \lambda^{-1}}\,. \tag{3.17}$$

3.2 The factor one-half of the action functional

To obtain the holomorphic anomaly equations (3.4) we used the Poisson algebra (2.6) but as is shown in section 3.1, the true algebra one has to use is the Dirac bracket algebra given by the eq. (3.16) which differs from the Poisson algebra by a factor one-half. Naively this makes a difference between the Schrödinger equations (3.2) and the holomorphic anomaly equations in topological string theory. This problem can be recovered by multiplying the right hand side of eq. (2.3) by a factor one-half.

Considering the action,

$$S = \alpha \int_{CY_3\times R} C\wedge dC\,, \tag{3.18}$$

one should modify eq. (3.7) slightly. For example,

$$p_i = i\alpha e^{-K} G_{i\bar{i}}\bar{x}^{\bar{i}}\,, \qquad \pi = -i\alpha e^{-K}\bar{\lambda}^{-1}\,, \tag{3.19}$$

which are assumed to satisfy the Poisson algebra (3.10). Although the constraints (3.9) also modify correspondingly, but one can show that the Dirac brackets (3.16) do not modify. Consequently, still after modifying the action one has,

$$\{x^i, p_j\}_{\rm DB} = \frac{\delta^i_j}{2}\,, \qquad \{\lambda^{-1}, \pi\}_{\rm DB} = \frac{1}{2}\,. \tag{3.20}$$

The generators H_i and $\bar{H}_{\bar{i}}$ given in eq. (2.12) should be modified in order to obtain eq. (2.11) using the modified equations of motion,

$$\begin{aligned} \mathcal{O}_{,i} &= \{\mathcal{O}, H_i\}_{\rm DB}\,, \\ \mathcal{O}_{,\bar{i}} &= \{\mathcal{O}, \bar{H}_{\bar{i}}\}_{\rm DB}\,. \end{aligned} \tag{3.21}$$

The modified H_i, $\bar{H}_{\bar{i}}$ are,

$$\begin{aligned} H_i &= -2\lambda^{-1}p_i - i\alpha C_{ijk}x^j x^k + \partial_i K[\pi,\lambda^{-1}]_+ + \Gamma^k_{ij}[x^j,p_k]_+ + \partial_i K[x^j,p_j]_+\,, \\ \bar{H}_{\bar{i}} &= -2G_{\bar{i}i}\pi x^i - \frac{i}{\alpha}e^{2K}\bar{C}_{\bar{i}\bar{j}\bar{k}}G^{\bar{j}j}G^{\bar{k}k}p_j p_k\,. \end{aligned} \tag{3.22}$$

One easily verifies that for Schrödinger equations (3.2) obtained by replacements (3.17) to be equivalent to the holomorphic anomalies (3.4) one has to impose,

$$\alpha = \frac{1}{2}\,. \tag{3.23}$$

Therefore the action functional for the quadratic field theory on $CY_3 \times R$ that gives the holomorphic anomalies in topological string theory is that given in eq. (1.3).

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